A Bi-directional Quantum Search Algorithm
-An Efficient Scalble Quantum Search Algorithm in the NISQ Era

Zain Hafeez, Debanjan Konar, and Vaneet Aggarwal

Purdue University, USA
vaneet@purdue.edu

March 3, 2025

Contents

Introduction to Quantum Search Algorithms

- Grover's quantum Search (GS) algorithm[1] is enabling quantum computers to perform a database search.

[1] Lov K. Grover, "Quantum mechanics helps in searching for a needle in a haystack," *Physical Review Letters*, vol. 79, no. 2, 1997, doi: `https://doi.org/10.1103/PhysRevLett.79.325`.

[2] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani, "Strengths and Weaknesses of Quantum Computing," *SIAM Journal on Computing*, vol. 26, no. 5, pp. 1510–1523, 1997, doi: `https://doi.org/10.1137/S0097539796300933`.

Introduction to Quantum Search Algorithms

- Grover's quantum Search (GS) algorithm[1] is enabling quantum computers to perform a database search.
- It quadratically outperform their classical counterparts with only $O(\sqrt{N})$ evaluations[2].

[1] Lov K. Grover, "Quantum mechanics helps in searching for a needle in a haystack," *Physical Review Letters*, vol. 79, no. 2, 1997, doi: https://doi.org/10.1103/PhysRevLett.79.325.

[2] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani, "Strengths and Weaknesses of Quantum Computing," *SIAM Journal on Computing*, vol. 26, no. 5, pp. 1510–1523, 1997, doi: https://doi.org/10.1137/S0097539796300933.

Introduction to Quantum Search Algorithms

- Grover's quantum Search (GS) algorithm[1] is enabling quantum computers to perform a database search.

- It quadratically outperform their classical counterparts with only $O(\sqrt{N})$ evaluations[2].

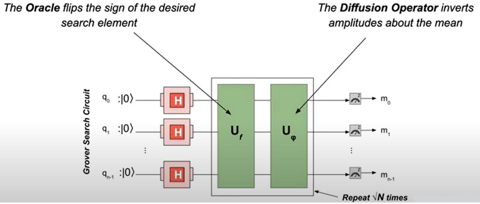


Figure 1: Grover's Iteration.

[1] Lov K. Grover, "Quantum mechanics helps in searching for a needle in a haystack," *Physical Review Letters*, vol. 79, no. 2, 1997, doi: `https://doi.org/10.1103/PhysRevLett.79.325`.

[2] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani, "Strengths and Weaknesses of Quantum Computing," *SIAM Journal on Computing*, vol. 26, no. 5, pp. 1510–1523, 1997, doi: `https://doi.org/10.1137/S0097539796300933`.

Introduction to Quantum Search Algorithms

- Grover's quantum Search (GS) algorithm[1] is enabling quantum computers to perform a database search.
- It quadratically outperform their classical counterparts with only $O(\sqrt{N})$ evaluations[2].

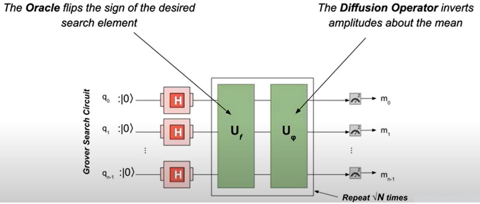


Figure 1: Grover's Iteration.

- However, classical GS is computationally expensive with an increase in the number of qubits or circuit depth.

[1] Lov K. Grover, "Quantum mechanics helps in searching for a needle in a haystack," *Physical Review Letters*, vol. 79, no. 2, 1997, doi: https://doi.org/10.1103/PhysRevLett.79.325.

[2] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani, "Strengths and Weaknesses of Quantum Computing," *SIAM Journal on Computing*, vol. 26, no. 5, pp. 1510–1523, 1997, doi: https://doi.org/10.1137/S0097539796300933.

Introduction to Partial Grover Search Algorithms

- The search often focuses on a block containing marked state than the entire dataset-Quantum Partial Search Algorithm (QPSA)[3].

[3] V. E. Korepin, and Lov K. Grover, "Simple Algorithm for Partial Quantum Search," *Quantum Inf. Proc.*, vol. 5, pp. 5–10, 2006, doi: `https://doi.org/10.1007/s11128-005-0004-z`.

[4] Lov K. Grover, and J. Radhakrishnan, "Is partial quantum search of a database any easier?," *Proc. of the 7th Annual ACM Symposium on Parallelism in Algorithms and Architectures*, pp. 186–194, 2005, doi: `10.1145/1073970.1073997`.

Introduction to Partial Grover Search Algorithms

- The search often focuses on a block containing marked state than the entire dataset-Quantum Partial Search Algorithm (QPSA)[3].
- Grover and Radhakrishnan (GRK)[4] proposed a faster QPSA using the same Oracle as the Grover algorithm.

[3] V. E. Korepin, and Lov K. Grover, "Simple Algorithm for Partial Quantum Search," *Quantum Inf. Proc.*, vol. 5, pp. 5–10, 2006, doi: https://doi.org/10.1007/s11128-005-0004-z.

[4] Lov K. Grover, and J. Radhakrishnan, "Is partial quantum search of a database any easier?," *Proc. of the 7th Annual ACM Symposium on Parallelism in Algorithms and Architectures*, pp. 186–194, 2005, doi: 10.1145/1073970.1073997.

Introduction to Partial Grover Search Algorithms

- The search often focuses on a block containing marked state than the entire dataset-Quantum Partial Search Algorithm (QPSA)[3].
- Grover and Radhakrishnan (GRK)[4] proposed a faster QPSA using the same Oracle as the Grover algorithm.
- A database with $\mathcal{N}$ items is divided into b identically sized blocks:

$$\mathcal{B} = \mathcal{N}/b \tag{1}$$

[3] V. E. Korepin, and Lov K. Grover, "Simple Algorithm for Partial Quantum Search," *Quantum Inf. Proc.*, vol. 5, pp. 5–10, 2006, doi: https://doi.org/10.1007/s11128-005-0004-z.

[4] Lov K. Grover, and J. Radhakrishnan, "Is partial quantum search of a database any easier?," *Proc. of the 7th Annual ACM Symposium on Parallelism in Algorithms and Architectures*, pp. 186–194, 2005, doi: 10.1145/1073970.1073997.

Introduction to Partial Grover Search Algorithms

- The search often focuses on a block containing marked state than the entire dataset-Quantum Partial Search Algorithm (QPSA)[3].
- Grover and Radhakrishnan (GRK)[4] proposed a faster QPSA using the same Oracle as the Grover algorithm.
- A database with $\mathcal{N}$ items is divided into b identically sized blocks:

$$\mathcal{B} = \mathcal{N}/b \tag{1}$$

- In QPSA, approximately $\frac{\pi}{4}(1 - c(b))\sqrt{\mathcal{N}}$ searches are required to locate the target block[4] [$c(b)$ is a correction factor].

[3] V. E. Korepin, and Lov K. Grover, "Simple Algorithm for Partial Quantum Search," *Quantum Inf. Proc.*, vol. 5, pp. 5–10, 2006, doi: https://doi.org/10.1007/s11128-005-0004-z.

[4] Lov K. Grover, and J. Radhakrishnan, "Is partial quantum search of a database any easier?," *Proc. of the 7th Annual ACM Symposium on Parallelism in Algorithms and Architectures*, pp. 186–194, 2005, doi: 10.1145/1073970.1073997.

Introduction to Partial Grover Search Algorithms

- GRK partial Grover's search randomly select $(b-1)$ blocks and apply GS iterations $\mathcal{N}(1-\frac{1}{b})$ inside the selected blocks.

- GRK partial Grover's search randomly select $(b-1)$ blocks and apply GS iterations $\mathcal{N}(1-\frac{1}{b})$ inside the selected blocks.

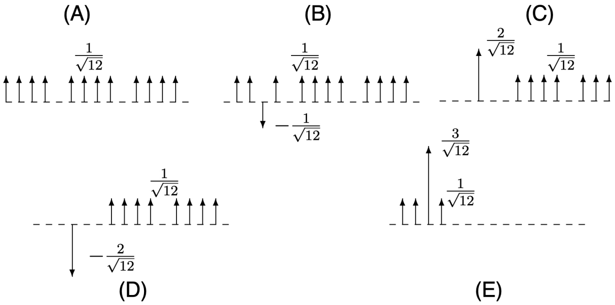


Figure 2: Partial Grover search[4] in a database of twelve items (A) Start with the uniform superposition of the twelve states (B) Invert the amplitude of the target state (C) Invert about the average in each of the three blocks (D) Invert the amplitude of the target state again (E) Invert about the global average.

- GRK partial Grover's search randomly select $(b-1)$ blocks and apply GS iterations $\mathcal{N}(1-\frac{1}{b})$ inside the selected blocks.

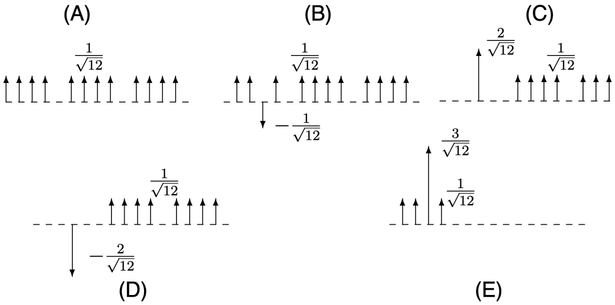


Figure 2: Partial Grover search[4] in a database of twelve items (A) Start with the uniform superposition of the twelve states (B) Invert the amplitude of the target state (C) Invert about the average in each of the three blocks (D) Invert the amplitude of the target state again (E) Invert about the global average.

- This would need $\frac{\pi}{4}\sqrt{\mathcal{B}(b-1)} = \frac{\pi}{4}\sqrt{\mathcal{N}}\sqrt{\frac{b-1}{b}} = \frac{\pi}{4}\sqrt{\mathcal{N}}\sqrt{1-\frac{1}{b}}$ iterations.

Limitation of Partial Grover's Search Algorithms

- Of late, Guo *et al.* introduced an improved version of QPSA using Depth-First Search (DFS) referred to as DFGS[5].

[5] H. Guo, "Depth-First Grover Search Algorithm on Hybrid Quantum-Classical Computer," *arXiv*, 2002, doi:https://doi.org/10.48550/arXiv.2210.04664.

[6] D. Konar, Z. Hafeez, and V. Aggarwal, "A Bi-directional Quantum Search Algorithm," *arXiv*, 2024, doi: https://arxiv.org/abs/2404.15616.

Limitation of Partial Grover's Search Algorithms

- Of late, Guo *et al.* introduced an improved version of QPSA using Depth-First Search (DFS) referred to as DFGS[5].
- However, despite its potential, QPSA have been limited in implementation due to their complex formulation.
- Our work provides a Bi-Directional Grover Search (BDGS) algorithm[6], which draws inspiration from the QPSA and DFGS.
- On parallelization, our novel Bi-directional approach requires $\frac{\pi}{4\sqrt{2}}\sqrt{\mathcal{N}}(1-\sqrt{\frac{1}{b^{r/2k}}})$ iterations (Here, $\mathcal{N}=2^r$, and $k=\lceil \log_2 b \rceil$)

[5] H. Guo, "Depth-First Grover Search Algorithm on Hybrid Quantum-Classical Computer," *arXiv*, 2002, doi:https://doi.org/10.48550/arXiv.2210.04664.

[6] D. Konar, Z. Hafeez, and V. Aggarwal, "A Bi-directional Quantum Search Algorithm," *arXiv*, 2024, doi: https://arxiv.org/abs/2404.15616.

Bi-directional Grover Search Algorithm

- A novel approach combining Bi-directional search (BDS) and Partial Grover Search (PGS) algorithms[7].

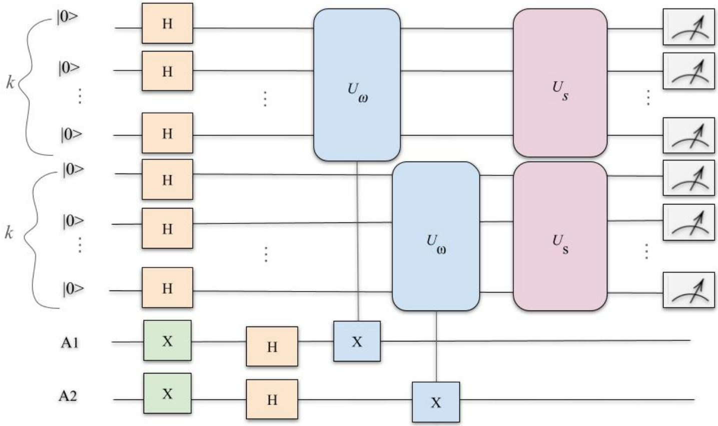


Figure 3: Quantum Circuit schematic for Bi-directional Grover's Quantum Search (U_ω and U_s are oracle and diffuser).

[7] R. E. Korf, "Search Techniques," *Encyclopedia of Information Systems*, pp. 31–43, 2003, doi: https://doi.org/10.1016/B0-12-227240-4/00155-6.

Bi-directional Grover Search Algorithm

- The proposed BDGS algorithm incorporates PGS[7] to determine the next $k = \lceil \log_2 b \rceil$ bits in forward and backward search in BDGS as

Bi-directional Grover Search Algorithm

- The proposed BDGS algorithm incorporates PGS[7] to determine the next $k = \lceil \log_2 b \rceil$ bits in forward and backward search in BDGS as

$$|\underbrace{X_0X_1\cdots X_iX_{i+1}}_{Found}\underbrace{\square\square}_{Next\ \ k}\cdots\rangle \xrightarrow[Forward]{\text{PGS}} |\underbrace{X_0X_1\cdots X_iX_{i+1}}_{Found}\underbrace{X_{i+2}X_{i+3}}_{Next\ \ k}\cdots\rangle$$

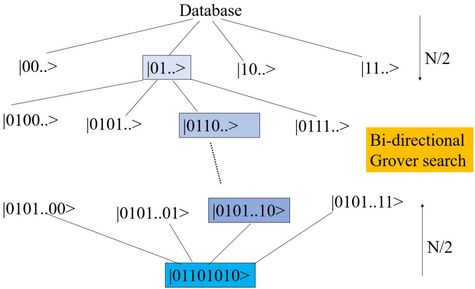
$$|\cdots\underbrace{\square\square}_{Prev.\ \ k}\underbrace{X_iX_{i+1}\cdots X_{r-1}X_r}_{Found}\rangle \xrightarrow[Backward]{\text{PGS}} |\cdots\underbrace{X_{i-2}X_{i-1}}_{Prev.\ \ k}\underbrace{X_iX_{i+1}\cdots X_{r-1}X_r}_{Found}\rangle \tag{2}$$

Bi-directional Grover Search Algorithm

- The proposed BDGS algorithm incorporates PGS[7] to determine the next $k = \lceil \log_2 b \rceil$ bits in forward and backward search in BDGS as

$$|\underbrace{X_0 X_1 \cdots X_i X_{i+1}}_{Found} \underbrace{\square\square}_{Next\ \ k} \cdots\rangle \xrightarrow[Forward]{\text{PGS}} |\underbrace{X_0 X_1 \cdots X_i X_{i+1}}_{Found} \underbrace{X_{i+2} X_{i+3}}_{Next\ \ k} \cdots\rangle$$

$$|\cdots \underbrace{\square\square}_{Prev.\ \ k} \underbrace{X_i X_{i+1} \cdots X_{r-1} X_r}_{Found}\rangle \xrightarrow[Backward]{\text{PGS}} |\cdots \underbrace{X_{i-2} X_{i-1}}_{Prev.\ \ k} \underbrace{X_i X_{i+1} \cdots X_{r-1} X_r}_{Found}\rangle \quad (2)$$

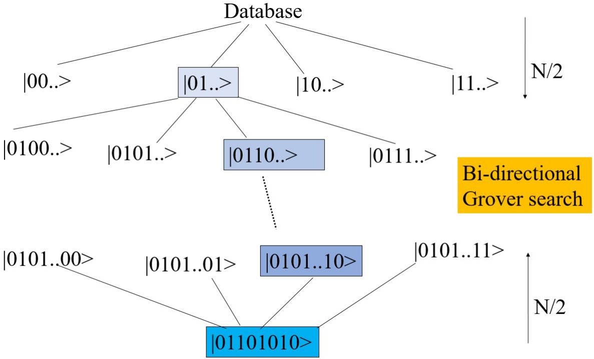


Bi-directional Grover Search Algorithm

- The proposed BDGS steps with $b = 4$ denote dividing the database into four parts for every layer with a PGS to get the next two bits.

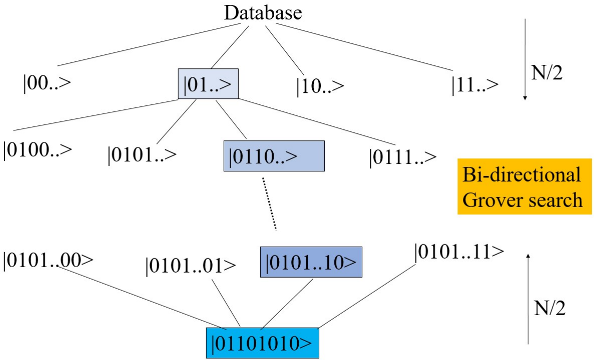
Bi-directional Grover Search Algorithm

- The proposed BDGS steps with $b = 4$ denote dividing the database into four parts for every layer with a PGS to get the next two bits.



Bi-directional Grover Search Algorithm

- The proposed BDGS steps with $b = 4$ denote dividing the database into four parts for every layer with a PGS to get the next two bits.



- The first four bits are encoded by the auxiliary qubits, which have the values $A_1 = 0110\cdots$ and $A_2 = 1111\cdots$.

BDGS: Computational Complexity Analysis

- We implemented the Oracle and Diffuser searching for $k = 2$ qubits.

BDGS: Computational Complexity Analysis

- We implemented the Oracle and Diffuser searching for $k = 2$ qubits.

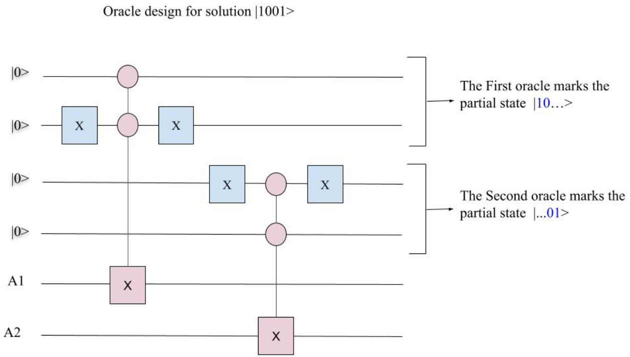


Figure 4: Design of Oracle for the proposed Bi-directional Grover Search for $b = 4$

BDGS: Computational Complexity Analysis

- We implemented the Oracle and Diffuser searching for $k = 2$ qubits.

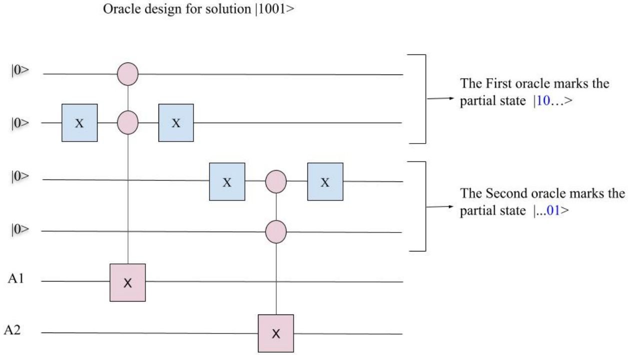
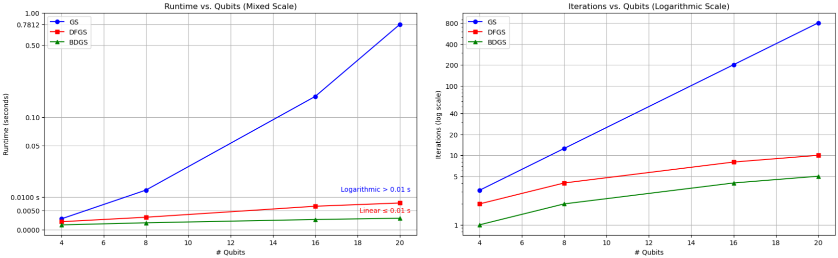


Figure 4: Design of Oracle for the proposed Bi-directional Grover Search for $b = 4$

- We deduce that the average number of oracle calls of BDGS is $\frac{\pi}{4\sqrt{2}}\sqrt{\mathcal{N}}(1 - \sqrt{\frac{1}{b^{r/2k}}})$.

BDGS: Simulation Results

- We have conducted various trials on quantum simulation on a qubit space ranging from 4 to 20 qubits on the Qiskit Aer simulator.



(a) **(b)**

Figure 5: Plot for (a) Runtime (s) vs. #Qubit (b) #Iteration vs. #Qubit search space for BDGS, DFGS [5], and standard

- With 1024 shots with each execution on 8-cores systems with a maximum frequency of 3.5 GHz and 8-GB RAM.

BDGS: Simulation Results

Table 1: Comparative performance analysis of the proposed Bi-directional Grover's Search (BDGS), standard Grover's Search (GS) [8], and Depth-First Grover's Search (DFGS) [5] with 1024 shots

		GS		DFGS		BDGS	
Qubits	**Trial**	**Acc.**	**Time(s)**	**Acc.**	**Time(s)**	**Acc.**	**Time(s)**
4	1	95.3	0.00267	100	0.00174	100	0.00117
	2	96.6	0.00245	100	0.00197	100	0.00113
	3	96.2	0.00222	100	0.00174	100	0.00108
	4	96.7	0.00263	100	0.00251	100	0.00104
	5	96.3	0.00300	100	0.00135	100	0.00104
	Avg.	96.02	0.00259	100	0.00186	100	0.00109
8	1	100	0.0148	100	0.00255	100	0.00147
	2	100	0.0127	100	0.00301	100	0.00165
	3	100	0.0131	100	0.00290	100	0.00170
	4	100	0.0123	100	0.00344	100	0.00152
	5	100	0.0133	100	0.00332	100	0.00163
	Avg.	100	0.0132	100	0.00304	100	0.00159
16	1	100	0.206	100	0.00676	100	0.00224
	2	100	0.152	100	0.00737	100	0.00256
	3	100	0.146	100	0.00617	100	0.00242
	4	100	0.150	100	0.00623	100	0.00248
	5	100	0.149	100	0.00589	100	0.00245
	Avg.	100	0.161	100	0.00645	100	0.00243
20	1	100	0.810	100	0.00708	100	0.00284
	2	100	0.755	100	0.00705	100	0.00284
	3	100	0.774	100	0.00799	100	0.00283
	4	100	0.780	100	0.00833	100	0.00261
	5	100	0.767	100	0.00786	100	0.00274
	Avg.	100	0.781	100	0.00766	100	0.00277

Discussions and Conclusions

- A novel attempt has been made to enable the Bi-directional implementations of Partial Grover's Search (PGS).

Discussions and Conclusions

- A novel attempt has been made to enable the Bi-directional implementations of Partial Grover's Search (PGS).
- The number of iterations for a single solution search for our BDGS, DFGS [5], and PGS [4] is evaluated as $\sqrt{\mathcal{N}}\left(1 - d\sqrt{\frac{1}{b^{\frac{r}{dk}}}}\right)$ with constraint $d \log d < \frac{r}{2}$ (d is the number of equal segments on r), $\sqrt{\mathcal{N}}\left(1 - \sqrt{\frac{1}{b^r}}\right)$, and $\frac{\pi}{4}\sqrt{\mathcal{N}}\sqrt{1 - \frac{1}{b}}$, respectively.
- We have shown that if b is large, $k = \log_2 b$ is also large and $\frac{r}{dk}$ is small for large d and hence, for some $r \geqslant \sqrt{dk}$,
$\left(1 - d\sqrt{\frac{1}{b^{\frac{r}{dk}}}}\right) \leqslant \left(1 - \sqrt{\frac{1}{b^r}}\right) \leqslant \sqrt{1 - \frac{1}{b}}$

Discussions and Conclusions

- A novel attempt has been made to enable the Bi-directional implementations of Partial Grover's Search (PGS).
- The number of iterations for a single solution search for our BDGS, DFGS [5], and PGS [4] is evaluated as $\sqrt{\mathcal{N}}\left(1-d\sqrt{\frac{1}{b^{\frac{r}{dk}}}}\right)$ with constraint $d\log d < \frac{r}{2}$ (d is the number of equal segments on r), $\sqrt{\mathcal{N}}\left(1-\sqrt{\frac{1}{b^r}}\right)$, and $\frac{\pi}{4}\sqrt{\mathcal{N}}\sqrt{1-\frac{1}{b}}$, respectively.
- We have shown that if b is large, $k=\log_2 b$ is also large and $\frac{r}{dk}$ is small for large d and hence, for some $r \geqslant \sqrt{dk}$, $\left(1-d\sqrt{\frac{1}{b^{\frac{r}{dk}}}}\right) \leqslant \left(1-\sqrt{\frac{1}{b^r}}\right) \leqslant \sqrt{1-\frac{1}{b}}$
- The proposed BDGS has the potential to extend to a multi-solution search in a hybrid quantum-classical setting.

Funding Organizations

References

[1] Lov K. Grover, "Quantum mechanics helps in searching for a needle in a haystack," *Physical Review Letters*, vol. 79, no. 2, 1997, doi: `https://doi.org/10.1103/PhysRevLett.79.325`.

[2] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani, "Strengths and Weaknesses of Quantum Computing," *SIAM Journal on Computing*, vol. 26, no. 5, pp. 1510–1523, 1997, doi: `https://doi.org/10.1137/S0097539796300933`.

[3] V. E. Korepin, and Lov K. Grover, "Simple Algorithm for Partial Quantum Search," *Quantum Inf. Proc.*, vol. 5, pp. 5–10, 2006, doi: `https://doi.org/10.1007/s11128-005-0004-z`.

[4] Lov K. Grover, and J. Radhakrishnan, "Is partial quantum search of a database any easier?," *Proc. of the Seventeenth Annual ACM Symposium on Parallelism in Algorithms and Architectures*, pp. 186–194, 2005, doi: `10.1145/1073970.1073997`.

[5] H. Guo, "Depth-First Grover Search Algorithm on Hybrid Quantum-Classical Computer," *arXiv*, 2002, doi:`https://doi.org/10.48550/arXiv.2210.04664`.

[6] R. E. Korf, "Search Techniques," *Encyclopedia of Information Systems*, pp. 31–43, 2003, doi: `https://doi.org/10.1016/B0-12-227240-4/00155-6`.

[7] D. Konar, Z. Hafeez, and V. Aggarwal, "A Bi-directional Quantum Search Algorithm," *arXiv*, 2024, doi: `https://arxiv.org/abs/2404.15616`.

[8] Lov K. Grover, "A fast quantum mechanical algorithm for database search," *In Proc. 28th Annu. ACM Symp. Theory Comput.*, vol. 79, no. 2, pp. 212–219, 1996, doi: `https://doi.org/10.1145/237814.237866`.

Thank You